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Research Article

Application of Complex Analysis to Series and Generalized Chybeshev Polynomials of First Kind

[1]Egahi Musa, [2]Agbata, B.C., [3]Omatola I.U.
[1]Department of Mathematics/Computer Science Benue State University, Makurdi, Nigeria
[2]Department of Mathematics University of Nigeria Nsukka, Nigeria
[3]Department of Mathematical Sciences, Kogi State University, Anyigba, Nigeria

Abstract

In this article, we considered application of complex analysis to series and generalized Chebyshev polynomials. DeMoivre's theorem and the general Binomial expansion were used to establish alternative method for generating Chebyshev polynomials. The usual triple recursion formula derived from trigonometric identities was used to generate Chebyshev polynomials which yields same results with the alternative method. Some illustrative examples were given to show ability of the method. The polynomials obtained can be used to demonstrate approximation where maximum component of error is minimized in numerical analysis.

Keywords: Chebysvev polynomials, Binomial theorem, DeMoivres' theorem, Complex Analysis, Series, trigonometry identity, Approximation

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Corresponding author: abcinfotech08@gmail.com

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Introduction

Chebyshev Polynomials are use in numerical analysis for approximating a function; a better approximation has minimum error. The generalized Chebyshev polynomials are named after Pafnuty Chebeyshev and they are set of orthogonal polynomials which have link with DeMoivres' theorem in complex analysis they can be derived recursively (Abramowitz and Stegun, 1983). The Chebyshev polynomials of first kind denoted T_n are sequence of orthogonal polynomials with the greatest possible principal coefficient subject to the condition that their absolute value on the closed interval [-1, 1] is bounded by 1. They also form extremal polynomials for many other properties (Dette, 1964). In theory of approximation, Chebyshev polynomials are very useful since the zeros of the Chebyshev polynomials of first kind T_n are also known as Chebyshev nodes because they are used as nodes of polynomial interpolation (Elliott, 1964). The main purpose of this article is to derive another method which can be used to obtain the so useful Chebyshev polynomials. Chebyshev (1821 – 1994), Russian Mathematician contributed to the theory of approximation and functions. This was later developed into the constructive theory of functions, by S. Bernstein (Abramowitz and Stegun, 1972).
For any integer n ≥ 0 we define Chebyshev Polynomials by

$$T_n(x) = cos(ncos^{-1}x), \qquad -1 \leq x \leq 1 \qquad (1.0)$$

Let

$$\theta = cos^{-1}x \ or \ x = cos\theta \qquad 0 \leq \theta \leq \pi \qquad (1.1)$$

Then

$$T_n(x) = cosn\theta, \quad n \geq 0 \qquad (1.2)$$

For example, if $n = 0$, $T_0(x) = cos(0.\theta) = 1$

$n = 1, \qquad T_1(x) = \cos(\theta) = x$

$n = 2, \qquad T_2(x) = cos(2\theta) = 2cos^2\theta - 1 = 2x^2 - 1$

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Triple Recursion Formula

Recall the trigonometric addition formula

$$cos\ (P \pm Q) = \ cosPcosQ \mp sinPsinQ \qquad (1.3)$$

For $n \geq 1$, $\ T_{n+1}(x) = cos[(n+1)\theta] = \cos(n\theta + \theta)$

$$= \cos(n\theta)\cos(\theta) - \sin(n\theta)\, sin\theta \qquad (1.4) \qquad T_{n-1}(x) =$$

$\cos[(n-1)\theta] = \cos(n\theta - \theta)$

$$= \ \cos(n\theta)\cos(\theta) + \sin(n\theta)\, sin\theta \qquad (1.5)$$

Add equations (1.4) and (1.5) we have

$$T_{n+1}(x) + T_{n-1}(x) = 2\cos(n\theta)\, cos\theta$$

Substitute $x\ and\ cosn\theta$ in equations (1.1) and (1.2) we obtain

$$T_{n+1}(x) + T_{n-1}(x) = 2xT_n(x)$$

$$T_{n+1}(x) = 2xT_n x - T_{n-1}(x) \qquad n \geq 1 \quad (1.6)$$

Equations (1.7) is called the triple recursion formula which can be used to generate Chebyshev polynomials instead of using equation (1.0). Here the coefficient of x^n in $T_n(x)$ is always2^{n-1}.

For example $\ T_0(x\ ) = 1 \qquad T_1(x) = x$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \qquad n \geq 1$$

For n = 2 $\quad T_3(x) = 2xT_2(x) - T_1(x) = 2x(x^2 - 1) - x = 4x^3 - 3x$

For n =3 $\quad T_4(x) = 2xT_3(x) - T_2(x) = 2x(4x^3 - 3x) - (\ 2x^2 - 1)$

$$= 8x^4 - 8x^2 + 1$$

Methods

DeMoivre's theorem States that for any n>0

$$\cos(n\theta) + isin(n\theta) = (cos\theta + isin\theta)^n \qquad (2.0)$$

Expanding (2.0) using binomial theorem, we have

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$$(cos\theta + isin\theta)^n = \binom{n}{0} cos^n\theta + \binom{n}{1} cos^{n-1}\theta(isin\theta) + \binom{n}{2} cos^{n-2}\theta(isin\theta)^2 + \ldots$$

$$+ \binom{n}{k} cos^{n-k}\theta(isin\theta)^k + \cdots + \binom{n}{n-1} cos\theta(isin\theta)^{n-1} + \binom{n}{n}(isin\theta)^n \qquad (2.1)$$

To obtain expansions for $\cos(n\theta)$ and $\sin(n\theta)$ we equate the real and imaginary parts,

$$\cos(n\theta) = cos^n\theta - \binom{n}{2} cos^{n-2}\theta sin^2\theta + \binom{n}{4} cos^{n-4}\theta sin^4\theta + \cdots \qquad (2.2)$$

$$\sin(n\theta) = \binom{n}{1} cos^{n-1}\theta sin\theta - \binom{n}{3} cos^{n-3}\theta sin^3\theta + \cdots \qquad (2.3)$$

The summation of equations (2.2) and (2.3) can be written as

$$\cos(n\theta) = \sum_{k=0}^{h} (-1)^k \binom{n}{2k} cos^{n-2k}\theta sin^{2k}\theta \qquad (2.4)$$

$$\sin(n\theta) = \sum_{k=0}^{p} (-1)^k \binom{n}{2k+1} cos^{n-2k-1}\theta sin^{2k+1}\theta \qquad (2.5)$$

where $\binom{n}{2k} = \frac{n(n-1)(n-2)\ldots(n-2k+1)}{(2k)!}$ $for\ k > 0$, $h = \frac{(2n-1)+(-1)^n}{4} = \left[\frac{n}{2}\right]$, $p = \left[\frac{n-1}{2}\right]$

The notation $[y]$ stands for the greatest integer less than or equal to the real number y while any function in the form $f(y) = [y]$ is known as the bracket function.For example$[0.34] = 0$ *and* $[1.004] = 1$. From equation (2.4)

$$\cos(n\theta) = \sum_{r=0}^{h} (-1)^r \binom{n}{2r} cos^{n-2r}\theta sin^{2r}\theta \qquad (2.6)$$

h = $\frac{(2n-1)+(-1)^n}{4}$ but $\sin^2\theta = 1 - cos^2\theta$

Equation (2.4) becomes

$$\cos(n\theta) = \sum_{r=0}^{h} (-1)^r \binom{n}{2r} cos^{n-2r}\theta(1 - cos^2\theta)^r \qquad (2.7)$$

$cosn\theta$ in thepolynomial form of $cos\theta = p_n(cosn\theta)$.

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Equation 2.7 is an nth degree polynomial. Suppose n is a positive even integer we apply binomial expansion to (1-$cos^2\theta$)^r and substituting into equation (2.7) to obtain

$$\cos(n\theta) = (-1)^h cos^{n-2h}\theta + \sum_{r=2}^{h+1} \frac{(-1)^{h+r-1}2^{2r-4}n}{r-1} \binom{\frac{n}{2}+r-2}{2r-3} cos^{n+2r-2h-2}\theta \quad (2.8)$$

Similarly if n is an odd integer, we have

$$\cos(n\theta) = \sum_{r=1}^{h+1} \frac{(-1)^{h+r-1}2^{2r-2}n}{2r-1} \binom{\frac{n}{2}+r-\frac{3}{2}}{2r-2} cos^{n+2r-2h-2}\theta \qquad (2.9)$$

For example, if $n = 6$, we obtained from equation (2.8), h = $\frac{(2.6)-1+(-1)^6}{4} = 3$, so that

$$cos6\theta = (-1)3cos^0\theta + \sum_{r=2}^{4} \frac{(-1)^{2+r}2^{2r-4}.6}{r-1} \binom{r+1}{2r-3} cos^{2r-2}\theta$$

$$= -1 + 6\binom{3}{1} cos^2\theta - 48cos^4\theta + 32cos^6\theta$$

$$= 32cos^6\theta - 48cos^4\theta + 18cos^2\theta - 1$$

Chebyshev Polynomials from Alternative Method

Here we defined generalised form of Chebyshev Polynomial as

$$T_n(x) = 2^{1-n}cosn\theta \quad n = 1,2,3 \ldots \qquad (3.0)$$

where x = $cos\theta$

From equation (3.0) we can establish expansions which can be used to generate special Chebyshev Polynomials using the value of $cosn\theta$ in equation (2.8).
From equation (3.0)

$$T_n(x) = 2^{1-n}cosn\theta \quad n = 1,2,3 \ldots$$

Substitute in the expansion for$cosn\theta$, when n is an even integer:

$$T_n(x) = 2^{1-n}[(-1)^k x^{n-2} + \sum_{r=2}^{k+1} \frac{(-1)^{k+r-1}2^{2r-4}n}{r-1} \binom{\frac{n}{2}+r-2}{2r-3} x^{n+2r-2k} \quad (3.1)$$

Since n is even, k = $\frac{2n-1+(-1)^n}{4} = \frac{2n}{4} = \frac{n}{2}$. Hence

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$$T_n(x) = 2^{1-n}(-1)^{\frac{n}{2}} + \sum_{r=2}^{\frac{n}{2}+1} \frac{2^{2r-4}2^{1-n}n(-1)^{\frac{n}{2}+r-1}}{r-1}\binom{\frac{n}{2}+r-2}{2r-3}x^{2r-2}$$

$$= 2^{1-n}(-1)^{\frac{n}{2}} + \frac{n}{2^{n+3}}\sum_{r=2}^{\frac{n}{2}+1} \frac{4^r(-1)^{\frac{n}{2}+r-1}}{r-1}\binom{\frac{n}{2}+r-2}{2r-3}x^{2r-2} \qquad (3.2)$$

Equation (3.2) can be used to generate Chebyshev Polynomials for positive even integers.

For example if $n = 2$ from equation (3.2) becomes

$$T_2(x) = [2^{-1}.(-1)] + \frac{\frac{2}{2^5}.4^2.(-1)^2\binom{1}{1}x^2}{2-1} = -\frac{1}{2} + x^2 = \frac{1}{2}(2x^2 - 1)$$

For $n = 4$ we obtain

$$T_4(x) = 2^{-3}(-1)^2 + \frac{4}{2^7}\sum_{r=2}^{3} \frac{4^r(-1)^{r+1}}{r-1}\binom{r}{2r-3}x^{2r-2}$$

$$= \frac{1}{8} + \frac{1}{2^5}\left[\frac{4^2.(-1)^3.}{2-1}\binom{2}{1}x^2 + \frac{4^3(-1)^4}{2}\binom{3}{3}x^4\right]$$

$$= \frac{1}{8} + \frac{1}{2^5}[-32x^2 + 32x^4] = \frac{1}{8}(8x^4 - 8x^2 + 1)$$

Again from equation (3.0) and (2.9)

$$Tn(x) = 2^{1-n}cosn\theta$$

$$= 2^{1-n}\sum_{r=2}^{k+1} \frac{(-1)^{k+r-1}n2^{2r-2}}{2r-1}\binom{\frac{n}{2}+r-\frac{3}{2}}{2r-2}x^{n+2r-2k-2} \qquad (3.3)$$

Since n is odd, k+1 = $\frac{2n-1+(-1)^n}{4} + 1 = \frac{n+1}{2}$and we have

$$T_n(x) = \frac{n}{2^{n+1}}\sum_{r=1}^{\frac{n+1}{2}} \frac{4^r(-1)^{\frac{n+1}{2}+r}}{2r-1}\binom{\frac{n}{2}+r-\frac{3}{2}}{2r-2}x^{2r-1} \qquad (3.4)$$

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Equation (3.4) can be used to compute

$$T_1(x) = \frac{1}{2^2}\frac{4.(-1)^2.}{2-1}\binom{0}{0}x = x$$

$$T_3(x) = \frac{3}{2^2}\sum_{r=1}^{2}\frac{4^r(-1)^{r+2}}{2r-1}\binom{r}{2r-2}x^{2r-1}$$

$$= \frac{3}{2}[-4x + \frac{16}{3}x^3 = \frac{1}{4}(4x^3 - 3x)$$

Conclusion

The purpose of this work as stated in the introduction is to establish an alternative method for generating Chebyshev polynomials other than the usual recursion formula and the ordinary generating function. From the results of our analysis, we conclude that polynomials generated from the alternative method are the same with those obtained when triple recursion formula and ordinary generating function were used. The polynomials have many uses in areas of applied mathematics, science and engineering such as waveform synthesis and trigonometry identities.

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